

Solution for 'Topics in complex analysis'

(10/12/2025)

**H 13.1 (The Cauchy-Riemann equations and  $\mathbb{C}$ -linearity on  $\mathbb{C}^n$ )**

Identify  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$  and let  $U \subset \mathbb{C}^n$ . Consider a differentiable function  $f : U \rightarrow \mathbb{C}$ . By definition, at each point  $a \in U$  there exists some  $\mathbb{R}$ -linear map  $Df(a) : \mathbb{R}^{2n} \rightarrow \mathbb{C}$  such that

$$\lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{|f(a+h) - f(a) - Df(a)h|}{\|h\|} = 0.$$

Show that  $Df(a)$  is  $\mathbb{C}$ -linear if and only if for all  $1 \leq j \leq n$  we have

$$\frac{\partial}{\partial \bar{z}_j} f(a) = 0,$$

where  $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$  and  $z = x + iy$  with  $x, y \in \mathbb{R}^n$ .

**Remark:** If  $f : U \rightarrow \mathbb{C}$  is only assumed to be  $C^1$  then it is differentiable (by standard real analysis), so the conclusion of the exercise holds. This proves the equivalence of Remark 9.2 (ii) and Definition 9.1. In fact one could even weaken the assumption that  $f$  is  $C^1$  to locally integrable (and  $f$  satisfies the Cauchy-Riemann equations in the sense of distributions), since the decomposition  $\Delta = 4 \sum_{j=1}^n \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}$  of the Laplacian operator gives  $\Delta f = 0$ , which by elliptic regularity implies that  $f$  can be corrected on a set of measure zero to become analytic, and this exercise applies.

**Solution H 13.1:**

We identify  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$  via  $x + iy \mapsto (x, y)$ . The  $\mathbb{C}$ -linearity of  $Df(a)$  means that for all  $\lambda = \lambda_1 + i\lambda_2$  with  $\lambda_1, \lambda_2 \in \mathbb{R}$  and all  $x, y \in \mathbb{R}^n$  we have

$$Df(a)((\lambda_1 + i\lambda_2)(x + iy)) = (\lambda_1 + i\lambda_2)Df(a)(x + iy).$$

Since  $Df(a)$  is  $\mathbb{R}$ -linear, this is equivalent to

$$Df(a)(-y, x) = iDf(a)(x, y)$$

for all  $x, y \in \mathbb{R}^n$ . Considering real and imaginary parts, we rewrite this as

$$\begin{aligned} \operatorname{Re}(Df(a)(-y, x)) &= -\operatorname{Im}(Df(a)(x, y)), \\ \operatorname{Im}(Df(a)(-y, x)) &= \operatorname{Re}(Df(a)(x, y)). \end{aligned}$$

By  $\mathbb{R}$ -linearity it is enough to check these equations along basis vectors. This leads to

$$\begin{aligned} \operatorname{Re} \left( \frac{\partial f}{\partial y_j}(a) \right) &= -\operatorname{Im} \left( \frac{\partial f}{\partial x_j}(a) \right), \\ \operatorname{Re} \left( -\frac{\partial f}{\partial x_j}(a) \right) &= -\operatorname{Im} \left( \frac{\partial f}{\partial y_j}(a) \right), \\ \operatorname{Im} \left( \frac{\partial f}{\partial y_j}(a) \right) &= \operatorname{Re} \left( \frac{\partial f}{\partial x_j}(a) \right), \\ \operatorname{Im} \left( -\frac{\partial f}{\partial x_j}(a) \right) &= \operatorname{Re} \left( \frac{\partial f}{\partial y_j}(a) \right) \end{aligned}$$

for all  $1 \leq j \leq n$ . Note that the first and fourth as well as the second and third equations are equivalent. Moreover, the first and second equations can be reformulated as

$$\frac{\partial f}{\partial x_j}(a) = \operatorname{Im} \left( \frac{\partial f}{\partial y_j}(a) \right) - i \operatorname{Re} \left( \frac{\partial f}{\partial y_j}(a) \right) = -i \frac{\partial f}{\partial y_j}(a).$$

This is clearly equivalent to  $\frac{\partial f}{\partial \bar{z}_j}(a) = 0$ . □

### H 13.2 (The slicing method in action)

In this exercise we transfer some well-known results from single-variable complex analysis to the multivariable setting. Show the following.

- a) Liouville's theorem: Every bounded entire function  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  is constant.
- b) Identity theorem: Let  $D \subset \mathbb{C}^n$  be a domain and  $f : D \rightarrow \mathbb{C}$  be holomorphic. If  $f|_{B_r(a)} \equiv 0$  for some  $a \in D$  and  $r > 0$ , then  $f \equiv 0$ .
- c) Open mapping theorem: Let  $D \subset \mathbb{C}^n$  be a domain and  $f : D \rightarrow \mathbb{C}$  be non-constant and holomorphic. Then  $f(D)$  is again a domain.
- d) Maximum principle: Let  $D \subset \mathbb{C}^n$  be a domain and  $f : D \rightarrow \mathbb{C}$  be holomorphic. If  $|f|$  attains its maximum on  $D$  then  $f$  is constant.

#### Solution H 13.2:

a) Let  $a, b \in \mathbb{C}^n$  and with the notation of Lemma 9.3 consider the slicing  $f_{a,b-a}$  which satisfies  $f_{a,b-a}(0) = f(a)$  and  $f_{a,b-a}(1) = f(b)$ . Note that  $f_{a,b-a}$  is holomorphic on  $\mathbb{C}$  and bounded, so that the standard Liouville's theorem implies that  $f_{a,b-a}(0) = f_{a,b-a}(1)$  which means that  $f(a) = f(b)$ . Since  $b \in \mathbb{C}^n$  was arbitrary we conclude that  $f$  is constant.

b) Define  $U = \{z \in D : f \equiv 0 \text{ in a neighborhood of } z\}$ . Then  $a \in U$ . Moreover,  $U$  is open. We claim that  $U$  is also closed in  $D$ . Then by connectedness of  $D$  it follows that  $U = D$ . Let  $z_n \rightarrow z \in D$  with  $z_n \in U$ . Assume that  $z \notin U$ . Then for each  $r > 0$  there exists  $w_r \in B_r(z) \cap D$  such that  $f(w_r) \neq 0$ . Let  $z_n$  be such that  $z_n \in B_r(z)$  for some  $r > 0$  such that  $B_{2r}(z) \subset D$ . Note that  $w_r \neq z_n$  since  $w_r \notin U$ . Then consider the slicing

$$f_{z_n, w_r - z_n} : D_{z_n, w_r - z_n} \rightarrow \mathbb{C}.$$

By convexity of  $B_r(z)$  it holds that  $[0, 1] \subset D_{z_n, w_r - z_n}$ . Since  $D_{z_n, w_r - z_n}$  is open, we further find a cylinder of the form  $Z_\delta := (-\delta, 1 + \delta) + i(-\delta, \delta) \subset \mathbb{C}$ , for  $\delta > 0$ , such that  $Z_\delta \subset D_{z_n, w_r - z_n}$ . Moreover, since  $z_n \in U$  it follows that there exists  $\tau_n > 0$  such that  $f_{z_n, w_r - z_n}|_{[0, \tau_n]} \equiv 0$ . Hence by the one-dimensional identity theorem it follows that  $f_{z_n, w_r - z_n}$  vanishes on  $Z_\delta$ . This implies the contradiction  $f(w_r) = f_{z_n, w_r - z_n}(1) = 0$ .

c) By continuity,  $f(D)$  is path-connected. Thus it only remains to prove that  $f(D)$  is open. Let  $w \in f(D)$  and consider  $z \in D$  such that  $f(z) = w$ . Consider a ball  $B_r(z) \subset D$ . Then  $f|_{B_r(z)}$  cannot be constant (by the identity theorem), so that there exists a point  $p \in B_r(z)$  such that  $f(p) \neq f(z)$ . In particular, the function  $f_{p, z-p} : D_{p, z-p} \rightarrow \mathbb{C}$  is not constant, so that by the one-dimensional open mapping theorem the set  $f_{p, z-p}(D_{p, z-p})$  contains an open neighborhood of  $f(z)$  (more precisely, we can consider  $f_{p, z-p}$  restricted to a suitable cylinder as in b) to have a domain where it is not constant). Since  $w \in f_{p, z-p}(D_{p, z-p}) \subset f(D)$  we deduce the claim.

d) If  $|f|$  attains its maximum on  $D$  at a point  $a$  and  $f$  is not constant, then there cannot exist a neighborhood of  $f(a)$  inside  $f(D) \subset \mathbb{C}$ , since  $|f(z)| \leq |f(a)|$  for all  $z \in D$ . This contradicts the open mapping theorem. □

### H 13.3 (Failure of the open mapping theorem in the fully vectorial case)

In H 13.2 we proved the open mapping theorem for functions  $f : D \rightarrow \mathbb{C}$ . Here we show that it is false for vectorial functions  $f : D \rightarrow \mathbb{C}^m$  for  $m \geq 2$  even when no component is constant. Define  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  by  $f(z_1, z_2) = (z_1, z_1 z_2)$ . Show that  $f$  is holomorphic but not an open map.

**Hint:** To guess where the map is not open, find where its differential is not invertible.

#### Solution H 13.3:

Observe that  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is holomorphic, since each component is a polynomial. Note that  $(0, 0) \in f(B_r(0))$  for any  $r > 0$ . Fix  $r > 0$ . We claim that there exists a sequence  $w_n \in \mathbb{C}^2$  such that  $w_n \rightarrow (0, 0)$  in  $\mathbb{C}^2$  and  $w_n \notin f(B_r(0))$  for all sufficiently large  $n \in \mathbb{N}$ . Indeed, define  $w_n = (1/n^2, 1/n)$ . Then  $f(z) = w_n$  if and only if  $z_1 = 1/n^2$  and  $z_2 = n$ . But then

$$\|(z_1, z_2)\| \geq n \cdot \|(0, 1)\| - \frac{1}{n^2} \cdot \|(1, 0)\| \rightarrow +\infty \quad \text{as } n \rightarrow \infty,$$

so that  $(z_1, z_2) \notin B_r(0)$  for  $n$  sufficiently large. Hence  $f(B_r(0))$  is not open. □

### H 13.4 (On power series in several variables)

a) Determine for each series below the largest open set  $U \subset \mathbb{C}^2$  where it converges absolutely. Is  $U$  convex?

(i)  $\sum_{n=0}^{\infty} z^n w^n,$  (ii)  $\sum_{n=1}^{\infty} z^n w^{n!}.$

b) Let  $F(z) = \sum_{\alpha \in (\mathbb{N}_0)^n} c_\alpha z^\alpha$  be a formal power series centered at the origin. Show that if  $z = (z_1, \dots, z_n)$  is such that  $F(z)$  converges absolutely, then  $F(\lambda_1 z_1, \dots, \lambda_n z_n)$  also converges absolutely if  $|\lambda_i| \leq 1$  for all  $1 \leq i \leq n$ .

#### Solution H 13.4:

a) (i) Since  $|z^n| \cdot |w^n| = |zw|^n$ , it follows from the corresponding result for geometric series that  $U = \{(z, w) \in \mathbb{C}^2 : |zw| < 1\}$ . This set is open, but not convex since the points  $(z, 0)$  and  $(0, w)$  belong to  $U$  for every choice of  $z, w \in \mathbb{C}$ , but the point  $(1, 1) := \frac{1}{2}(2, 0) + \frac{1}{2}(0, 2)$  does not belong to  $U$ .

(ii) Here the set  $U$  is given by  $U = \{(z, w) \in \mathbb{C}^2 : |w| < 1\}$ . Indeed, first observe that this set is open. Next, for any fixed  $(z, w) \in U$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  it holds that

$$|z^n w^{n!}| = |z|^n \cdot |w|^{n!} \leq (|z| \cdot |w|^{n-1})^n \leq \frac{1}{2^n},$$

where we used that  $|w|^{n-1} \rightarrow 0$  as  $n \rightarrow +\infty$ . Hence the series converges for all points in  $U$ . Next consider a point  $(z, w) \in \mathbb{C}^2$  such that  $|w| \geq 1$ . Then any neighborhood of  $(z, w)$  contains a point  $(z', w')$  such that  $z' \neq 0$  and  $|w'| > 1$ . Then  $((z')^{-1}, (w')^{-1}) \in U$  and from the above estimate we deduce that  $|(z')^n (w')^{n!}| \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Hence the power series cannot converge in  $(z', w')$ , so that the point  $(z, w)$  cannot belong to the largest open set where the power series converges. Here the set  $U$  is clearly convex.

**Remark:** Note that the series (ii) converges also on the sets  $0 \times \mathbb{C}$  and  $B_1(0) \times \partial B_1(0)$ , but those add no interior points.

b) Note that

$$\left| c_\alpha \prod_{i=1}^n (\lambda_i z_i)^{\alpha_i} \right| = |c_\alpha| \cdot \prod_{i=1}^n |\lambda_i|^{\alpha_i} |z_i|^{\alpha_i} \leq |c_\alpha| \cdot \prod_{i=1}^n |z_i|^{\alpha_i} = \left| c_\alpha \prod_{i=1}^n z_i^{\alpha_i} \right|,$$

so that the absolute convergence of  $F(z)$  implies the absolute convergence of  $F(\lambda_1 z_1, \dots, \lambda_n z_n)$ .

**Remark:** Note that this property implies that the domain of convergence is simply connected, since any closed curve inside it can be contracted to the origin.

□